

ADDENDUM TO: Generically split projective homogeneous varieties

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Abstract

In this addendum we generalize some results of our article [PS10]. More precisely, we remove all restrictions on the characteristic of the base field (in [PS10] we assumed that the characteristic is different from any torsion prime of the group), and complete our classification by the last missing case, namely PGO_{2n}^+ . We follow our notation from [PS10].

1 Chow rings of reductive groups

1.1. Let G_0 be a split reductive algebraic group defined over a field k . We fix a split maximal torus T in G_0 and a Borel subgroup B of G_0 containing T and defined over k . We denote by Φ the root system of G_0 , by Π the set of simple roots of Φ with respect to B , and by \widehat{T} the group of characters of T . Enumeration of simple roots follows Bourbaki.

Any projective G_0 -homogeneous variety X is isomorphic to G_0/P_Θ , where P_Θ stands for the (standard) parabolic subgroup corresponding to a subset $\Theta \subset \Pi$. As P_i we denote the maximal parabolic subgroup $P_{\Pi \setminus \{\alpha_i\}}$ of type i .

Consider the characteristic map $c: S(\widehat{T}) \rightarrow \mathrm{CH}^*(G_0/B)$ from the symmetric algebra of \widehat{T} to the Chow ring of G_0/B given in [PS10, 2.7], and denote its image by R^* . According to [Gr58, Rem. 2°], the ring $\mathrm{CH}^*(G_0)$ can

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be presented as the quotient of $\mathrm{CH}^*(G_0/B)$ modulo the ideal generated by the non-constant elements of R^* .

1.2 Lemma. *The pull-back map*

$$\mathrm{CH}^*(G_0) \rightarrow \mathrm{CH}^*([G_0, G_0])$$

is an isomorphism.

Proof. Indeed, $B' = B \cap [G_0, G_0]$ is a Borel subgroup of $[G_0, G_0]$, the map

$$[G_0, G_0]/B' \rightarrow G_0/B$$

is an isomorphism, and the map $S(\widehat{T}) \rightarrow \mathrm{CH}^*(G_0/B)$ factors through the surjective map $S(\widehat{T}) \rightarrow S(\widehat{T}')$, where $T' = T \cap [G_0, G_0]$. \square

Let P be a parabolic subgroup of G_0 . Denote by L the Levi subgroup of P and set $H_0 = [L, L]$. We have

1.3 Lemma. *The pull-back map*

$$\mathrm{CH}^*(P) \rightarrow \mathrm{CH}^*(H_0)$$

is an isomorphism.

Proof. The quotient map $P \rightarrow L$ is Zariski locally trivial affine fibration, therefore the pull-back map $\mathrm{CH}^*(L) \rightarrow \mathrm{CH}^*(P)$ is an isomorphism. Since the composition $L \rightarrow P \rightarrow L$ is the identity map, the pull-back map $\mathrm{CH}^*(P) \rightarrow \mathrm{CH}^*(L)$ is an isomorphism as well. It remains to apply Lemma 1.2. \square

1.4 Lemma. *The pull-back map*

$$\mathrm{CH}^*(G_0) \rightarrow \mathrm{CH}^*(P)$$

is surjective.

Proof. Applying [Gr58, Proposition 3] to the natural map $G_0/B \rightarrow G_0/P$ we see that the map $\mathrm{CH}^*(G_0/B) \rightarrow \mathrm{CH}^*(P/B)$ is surjective. But the map $\mathrm{CH}^*(P/B) \rightarrow \mathrm{CH}^*(P)$ is also surjective by Lemma 1.3 and fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^*(G_0/B) & \twoheadrightarrow & \mathrm{CH}^*(P/B) \\ \downarrow & & \downarrow \\ \mathrm{CH}^*(G_0) & \longrightarrow & \mathrm{CH}^*(P). \end{array}$$

\square

1.5 (Definition of σ). Now we restrict to the situation when G_0 is simple. Let p be a prime integer. Denote $\text{Ch}^*(-)$ the Chow ring with \mathbb{F}_p -coefficients. Explicit presentations of the Chow rings with \mathbb{F}_p -coefficients of split semisimple algebraic groups are given in [Kc85, Theorem 3.5].

For G_0 and H_0 they look as follows:

$$\text{Ch}^*(G_0) = \mathbb{F}_p[x_1, \dots, x_r] / (x_1^{p^{k_1}}, \dots, x_r^{p^{k_r}}) \text{ with } \deg x_i = d_i, 1 \leq d_1 \leq \dots \leq d_r;$$

$$\text{Ch}^*(H_0) = \mathbb{F}_p[y_1, \dots, y_s] / (y_1^{p^{l_1}}, \dots, y_s^{p^{l_s}}) \text{ with } \deg y_m = e_m, 1 \leq e_1 \leq \dots \leq e_s$$

for some integers k_i , l_i , d_i , and e_i depending on the Dynkin types of G_0 and H_0 .

By the previous lemmas the pull-back $\varphi: \text{Ch}^*(G_0) \rightarrow \text{Ch}^*(H_0)$ is surjective. For a graded ring S^* denote by S^+ the ideal generated by the non-constant elements of S^* . The induced map

$$\text{Ch}^+(G_0) / \text{Ch}^+(G_0)^2 \rightarrow \text{Ch}^+(H_0) / \text{Ch}^+(H_0)^2$$

is also surjective. Moreover, for any m with $e_m > 1$ there exists a unique i such that $d_i = e_m$. We denote $i =: \sigma(m)$. The surjectivity implies that

$$\varphi(x_{\sigma(m)}) = cy_m + \text{lower terms}, \quad c \in \mathbb{F}_p^\times.$$

2 Generically split varieties

For a semisimple group G and a prime number p denote by

$$J_p(G) = (j_1(G), \dots, j_r(G))$$

its J -invariant defined in [PSZ08].

2.1 Theorem. *Let G_0 be a split simple algebraic group over k , $G = {}_\gamma G_0$ be the twisted form of G_0 given by a 1-cocycle $\gamma \in H^1(k, G_0)$, $X = {}_\gamma(G_0/P)$ be the twisted form of G_0/P , and $Y = {}_\gamma(G_0/B)$ be the twisted form of G_0/B . The following conditions are equivalent:*

1. X is generically split;
2. The composition map

$$\overline{\text{CH}}^*(Y) \rightarrow \text{CH}^*(G_0) \rightarrow \text{CH}^*(P)$$

is surjective;

3. For every prime p the composition map

$$\overline{\text{Ch}}^1(Y) \rightarrow \text{Ch}^1(G_0) \rightarrow \text{Ch}^1(P)$$

is surjective, and

$$j_{\sigma(m)}(G) = 0 \text{ for all } m \text{ with } d_m > 1.$$

Proof. 1 \Rightarrow 2. The same argument as in the proof of Lemma 1.4 (with Y instead of G_0/B and X instead of G_0/P).

2 \Rightarrow 3. Clearly, the composition

$$\overline{\text{Ch}}^*(Y) \rightarrow \text{Ch}^*(G_0) \rightarrow \text{Ch}^*(P)$$

is surjective for every p . In particular, when $d_m > 1$ $\overline{\text{Ch}}^{d_m}(Y)$ contains an element of the form $x_{\sigma(m)} + a$, where a is decomposable, hence $j_{\sigma(m)}(G) = 0$.

3 \Rightarrow 1. $G_{k(X)}$ has a parabolic subgroup of type P ; denote the derived group of its Levi subgroup by H . We want to prove that H is split. By [PS10, Proposition 3.9(3)] it suffices to show that $J_p(H)$ is trivial for every p .

Denote the variety of complete flags of H by Z . It follows from the commutative diagram

$$\begin{array}{ccc} \text{Ch}^*(Y_{k(X)}) & \longrightarrow & \text{Ch}^*(Z) \\ \downarrow & & \downarrow \\ \text{Ch}^*(\overline{G}) & \longrightarrow & \text{Ch}^*(\overline{H}) \end{array}$$

that $j_m(H) \leq j_{\sigma(m)}(G)$ if $d_m > 1$. Therefore

$$j_m(H) \leq j_{\sigma(m)}(G_{k(X)}) \leq j_{\sigma(m)}(G) = 0$$

when $d_m > 1$. It remains to show that $\text{Ch}^1(\overline{Z})$ is rational. But this follows from the commutative diagram

$$\begin{array}{ccccc} \text{Ch}^1(Y) & \longrightarrow & \text{Ch}^1(Y_{k(X)}) & \longrightarrow & \text{Ch}^1(Z) \\ & & \downarrow & & \downarrow \\ & & \text{Ch}^1(\overline{G}) & \longrightarrow & \text{Ch}^1(\overline{H}) = \text{Ch}^1(P). \end{array}$$

□

2.2 Remark.

- If all $e_m > 1$, then the condition on $\overline{\text{Ch}}^1(Y)$ is void.
- If G_0 is different from PGO_{2n}^+ and $e_1 = 1$ (resp. $G_0 = \text{PGO}_{2n}^+$ and $e_1 = e_2 = 1$), then in view of [PS10, Proposition 4.2] it is equivalent to the fact that all Tits algebras of G are split. The latter is also equivalent to the fact that $j_1(G) = 0$ (resp. $j_1(G) = j_2(G) = 0$).
- If $G_0 = \text{PGO}_{2n}^+$ and there is exactly one m with $e_m = 1$, then there are exactly two fundamental weights among $\bar{\omega}_1, \bar{\omega}_{n-1}, \bar{\omega}_n$ whose image with respect to the composition $\text{Ch}^1(\bar{Y}) \rightarrow \text{Ch}^1(\bar{G}) \rightarrow \text{Ch}^1(\bar{H})$ equals y_1 . Then the condition on $\overline{\text{Ch}}^1(Y)$ is equivalent to the fact that at least one of the Tits algebras corresponding to these fundamental weights in the preimage of y_1 is split.

For a simple group G we denote by A_i its Tits algebra corresponding to $\bar{\omega}_i$.

2.3 Theorem. *Let G be a group given by a 1-cocycle from $H^1(k, G_0)$, where G_0 stands for the split adjoint group of the same type as G , and let X be the variety of the parabolic subgroups of G of type i .*

The variety X is generically split if and only if

G_0	i	conditions on G
PGL_n	any i	$\gcd(\exp A_1, i) = 1$
PGSp_{2n}	any i	i is odd or G is split
O_{2n+1}^+	any i	$j_m(G) = 0$ for all $1 \leq m \leq \lfloor \frac{n+1-i}{2} \rfloor$
PGO_{2n}^+	i is odd, $i < n-1$	$[A_{n-1}] = 0$ or $[A_n] = 0$, and $j_m(G) = 0$ for all $2 \leq m \leq \lfloor \frac{n+2-i}{2} \rfloor$
PGO_{2n}^+	i is even, $i < n-1$	$j_m(G) = 0$ for all $1 \leq m \leq \lfloor \frac{n+2-i}{2} \rfloor$
PGO_{2n}^+	$i = n-1$ or $i = n$, n is odd	none
PGO_{2n}^+	$i = n-1$, n is even	$[A_1] = 0$ or $[A_n] = 0$
PGO_{2n}^+	$i = n$, n is even	$[A_1] = 0$ or $[A_{n-1}] = 0$
E_6	$i = 3, 5$	none
E_6	$i = 2, 4$	$J_3(G) = (0, *)$
E_6	$i = 1, 6$	$J_2(G) = (0)$
E_7	$i = 2, 5$	none
E_7	$i = 3, 4$	$J_2(G) = (0, *, *, *)$

E_7	$i = 6$	$J_2(G) = (0, 0, *, *)$ ($J_2(G) = (0, 0, 0, 0)$ if $\text{char } k \neq 2$)
E_7	$i = 1$	$J_2(G) = (0, 0, 0, *)$ ($J_2(G) = (0, 0, 0, 0)$ if $\text{char } k \neq 2$)
E_7	$i = 7$	$J_3(G) = (0)$ and $J_2(G) = (*, 0, *, *)$ ($J_2(G) = (*, 0, 0, 0)$ if $\text{char } k \neq 2$)
E_8	$i = 2, 3, 4, 5$	<i>none</i>
E_8	$i = 6$	$J_2(G) = (0, *, *, *)$ ($J_2(G) = (0, 0, 0, *)$ if $\text{char } k \neq 2$)
E_8	$i = 1$	$J_2(G) = (0, 0, *, *)$ ($J_2(G) = (0, 0, 0, *)$ if $\text{char } k \neq 2$)
E_8	$i = 7$	$J_3(G) = (0, *)$ and $J_2(G) = (0, *, *, *)$ ($J_3(G) = (0, 0)$ if $\text{char } k \neq 3$, $J_2(G) = (0, 0, 0, *)$ if $\text{char } k \neq 2$)
E_8	$i = 8$	$J_3(G) = (0, *)$ and $J_2(G) = (0, 0, 0, *)$ ($J_3(G) = (0, 0)$ if $\text{char } k \neq 3$)
F_4	$i = 1, 2, 3$	<i>none</i>
F_4	$i = 4$	$J_2(G) = (0)$
G_2	<i>any</i> i	<i>none</i>

(“*” means “any value”).

Proof. Follows immediately from Theorem 2.1 and [PSZ08, Table 4.13]. \square

This theorem allows to give a shortened proof of the main result of [Ch10]:

2.4 Corollary. Let G be a group of type E_8 over a field k with $\text{char } k \neq 3$. If the 3-component of the Rost invariant of G is zero, then G splits over a field extension of degree coprime to 3.

Proof. Let K/k be a field extension of degree coprime to 3 such that the 2-component of the Rost invariant of G_K is zero.

Consider the variety X of parabolic subgroups of G_K of type 7. The Rost invariant of the semisimple anisotropic kernel of $G_{K(X)}$ is zero. Therefore $G_{K(X)}$ splits, and, thus, X is generically split.

By Theorem 2.3 $J_3(G_K) = (0, 0)$, hence by [PS10, Proposition 3.9(3)] G_K splits over a field extension of degree coprime to 3. This implies the corollary. \square

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